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Maximal Acceleration, Maximal Angular Velocity, and Causal Influence

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The upper bounds to acceleration and angular velocity which are suggested by quantum gravitational effects are described, together with the relativistic bound to velocity, by means of a cone $\mathcal{T}^+$ in the Lie algebra of the Poincaré group. The connection between these bounds and the existence of a minimal measurable length (of the order of Planck's length) is illustrated by means of a simple model. The geometric properties of the cone $\mathcal{T}^+$ and of other related structures are examined in some detail. The new geometric background requires some modifications of the concepts of causal influence and of spacetime coincidence, which are analyzed and shown to lead to some nonlocal features of the theory. Due to the smallness of Planck's length, these modifications to the causal relations cannot be observed by means of available experimental methods, but they could have some influence on the structure of elementary particles and on the very early cosmology.

1. INTRODUCTION

It has been suggested by Caianiello (1981), Caianiello *et al.* (1982), and Toller (1977, 1981) that the geometric background of a physical theory should take into account, besides the relativistic upper limit c to the velocity, also an upper bound to the acceleration of material bodies. More recently Brandt (1983, 1987) found that an upper bound of the order of $c^2 l_P^{-1}$, where l_P is the Planck length, is expected as a consequence of quantum gravitational effects. We shall see that there is a connection between the upper bounds to acceleration and to angular velocity and a limit of the order of l_P to the extension of physical objects or to the accuracy of position measurements, which also is a quantum gravitational effect, as shown by Ferretti (1984).

In the present paper we develop the geometric aspects of a formalism proposed by Toller (1981), in which the upper bounds to velocity, angular

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velocity, and acceleration are described by a convex cone $\mathscr{T}^+$ in the Lie algebra $\mathscr{T}$ of the Poincaré group $\mathscr{P}$. We introduce in this Lie algebra a basis composed of the generators A_k of the spacetime translations and the generators $A_{rs} = -A_{sr}$ of the homogeneous Lorentz transformations. A generic element $B \in \mathscr{T}$ can be written in the form

$$B = b^k A_k + \tfrac{1}{2} b^{rs} A_{rs} = b^\alpha A_\alpha \tag{1.1}$$

where α stands for k or for rs. We use also the vector notation

$$\mathbf{b} = (b^1, b^2, b^3), \qquad \mathbf{b}' = (b^{23}, b^{31}, b^{12}), \qquad \mathbf{b}'' = (b^{10}, b^{20}, b^{30}) \tag{1.2}$$

We indicate by $\mathscr{S}$ the 10-dimensional space of the pseudo-orthonormal coordinate frames in Minkowski space. The Poincaré group $\mathscr{P}$ acts transitively on $\mathscr{S}$ and if we choose an element $s_0 \in \mathscr{S}$, the mapping $g \to g s_0$ is a diffeomorphism of $\mathscr{P}$ and $\mathscr{S}$. We consider A_k and A_{rs} as right invariant vector fields on $\mathscr{P}$; they define vector fields on $\mathscr{S}$, which we indicate by the same symbols, and which do not depend on the choice of s_0. Note that all the tangent spaces of $\mathscr{P}$ and of $\mathscr{S}$ can be identified with $\mathscr{T}$: in fact, an infinitesimal displacement of a point in $\mathscr{S}$ can be considered as an infinitesimal Poincaré transformation of a coordinate frame in Minkowski space, described by an element of the form (1.1) of the Lie algebra $\mathscr{T}$. If we consider a smooth line $\lambda \to s(\lambda) \in \mathscr{S}$ parametrized by the variable λ and we put

$$\frac{ds(\lambda)}{d\lambda} = b^k A_k + \tfrac{1}{2} b^{rs} A_{rs} \tag{1.3}$$

then we find that the quantities

$$\mathbf{v} = \frac{\mathbf{b}}{b^0}, \qquad \boldsymbol{\omega} = \frac{\mathbf{b}'}{b^0}, \qquad \mathbf{a} = \frac{\mathbf{b}''}{b^0} \tag{1.4}$$

are, respectively, the components of the velocity, the angular velocity, and the acceleration of the moving frame $s(\lambda)$ measured with respect to the frame $s(\lambda)$ itself. A convex cone $\mathscr{T}^+ \subset \mathscr{T}$ describes a limitation on these three quantities.

For instance, we can consider a rigid body which defines a moving coordinate frame $s(\lambda)$ in which it is at rest. This frame follows a trajectory in $\mathscr{S}$ which describes the motion of the body, as discussed, for instance, by Hanson and Regge (1974). Since we are dealing with rest frames, we have $\mathbf{v} = 0$. It has been shown by Toller (1977, 1978, 1981) that infinitesimal transformations with $\mathbf{v} \neq 0$ become relevant when one studies the feasibility of the construction of a new reference frame starting from a preexistent one. Reference frames are defined by material objects and are built by means of physical operations which are feasible only if the Poincaré transformation which connects the two frames has certain properties. The infinitesimal feasible transformations are described by elements of the Lie algebra belonging to $\mathcal{T}^+$. In the presence of a gravitational field, the space $\mathcal{S}$ of the local inertial frames (tetrads) is the bundle of the pseudo-orthogonal frames of a pseudo-Riemannian spacetime $\mathcal{M}$ and the vector fields A_k have to be interpreted as infinitesimal parallel displacements of the local frames.

In the next section we discuss a naive model which illustrates the connection between the upper bounds to acceleration and angular velocity and the existence of a minimal extension of material bodies, which is briefly discussed in Appendix A. In Section 3 we summarize the main properties of the cone $\mathcal{T}^+$ and of its symmetry group $GL(4, R)$; a simplified proof of the uniqueness of $\mathcal{T}^+$ is given in Appendix B. In Section 4 we discuss the concept of causal influence on the basis of the analysis of $SL(4, R)$ covariant free quantum fields given in Toller (1988). We distinguish between the strong causal influence, which is a transitive relation, and the weak causal influence, which does not have this property, and we show by means of an example that this distinction is very natural in any theory based on extended objects. We discuss also the connection between the concepts of reciprocal causal influence and of spacetime coincidence.

In Section 5 we give a detailed treatment of the equations which define $\mathcal{T}^+$ and other geometric structures in the space $\mathcal{T}$. In Section 6 we apply the ideas of Section 4 to the bundle of the orthonormal frames of Minkowski spacetime. In Section 7 we summarize the conclusions and we show that, as a consequence of the smallness of the Planck length, the causal structures examined in the present paper cannot be distinguished from the causal structures of the usual relativistic theories by means of available experimental methods.

2. A SIMPLE MODEL FOR MAXIMAL ACCELERATION AND ANGULAR VELOCITY

It has been shown by Ferretti (1984) that quantum gravitational effects impose a limit l of the order of Planck's length to the accuracy of a position measurement and it follows that it is not meaningful to consider an object confined in a sphere or radius smaller than l. A more direct argument is given in Appendix A. It is clear that a spherical rigid body of radius r cannot have an angular velocity larger than c/r, otherwise some of its points would have a velocity larger than c. A similar reasoning shows that the body cannot have an acceleration larger than c^2/r. Since we must have $r \geq l$, we see that quantum gravitational effects give rise to an upper bound to the acceleration and the angular velocity of a material object.

In the present section we show that this simple argument, if developed with more detail, gives an upper bound to a particular expression containing both the vectors $\boldsymbol{\omega}$ and $\mathbf{a}$. This argument has a limited validity, because it concerns the velocity of small parts of a body which has already the minimal possible extension. However, we shall see in the next section that the same inequality follows from general symmetry arguments and the naive treatment given here can be useful to help the geometric intuition.

As mentioned in the preceding section, one can associate with an extended body a moving frame $s(\lambda)$ with respect to which it is at rest at every time, namely for every value of the parameter λ. The trajectory of a given point of the extended particle can be parametrized by the variable λ and it is composed of points with constant coordinates $(0, \mathbf{x})$ with respect to the moving reference frame $s(\lambda)$. The coordinates x'^k of the point corresponding to the value $\lambda + d\lambda$ of the parameter with respect to the reference frame $s(\lambda)$ can be obtained by means of the infinitesimal Poincaré transformation

$$x'^k = x^k + (b^k + b^{ik} x_i)\, d\lambda \tag{2.1}$$

with $\mathbf{b} = 0$ applied to $(0, \mathbf{x})$, namely they are given by

$$\begin{aligned} x'^0 &= (b^0 + \mathbf{b}'' \cdot \mathbf{x})\, d\lambda = (1 + \mathbf{a} \cdot \mathbf{x}) b^0\, d\lambda \\ \mathbf{x}' &= \mathbf{x} + \mathbf{b}' \times \mathbf{x}\, d\lambda = \mathbf{x} + \boldsymbol{\omega} \times \mathbf{x} b^0\, d\lambda \end{aligned} \tag{2.2}$$

This point has a velocity smaller than $c = 1$ if $x'^0 - x^0 \geq \|\mathbf{x}' - \mathbf{x}\|$, namely if

$$\|\boldsymbol{\omega} \times \mathbf{x}\| \leq 1 + \mathbf{a} \cdot \mathbf{x} \tag{2.3}$$

Since this inequality must hold for any value of $\mathbf{x}$ with $\|\mathbf{x}\| \leq l$, from a simple calculation we obtain the condition

$$\boldsymbol{\omega}^2 + \mathbf{a}^2 + 2\|\mathbf{a} \times \boldsymbol{\omega}\| \leq l^{-2} \tag{2.4}$$

In the following we use l as the unit of length, namely we put $l = 1$. The condition (2.4) can also be written in the form

$$b^0 \geq (\mathbf{b}'^2 + \mathbf{b}''^2 + 2\|\mathbf{b}' \times \mathbf{b}''\|)^{1/2} \tag{2.5}$$

3. DESCRIPTION OF THE CONE $\mathcal{T}^+$

It has been shown by Toller (1981) that the cone $\mathcal{T}^+ \subset \mathcal{T}$ is uniquely determined up to the choice of the unit of length and of the time direction by the following conditions:

(A) $\mathcal{T}^+$ is a wedge, namely it is convex and invariant under dilatations.
(B) It is a cone, namely a wedge which does not contain straight lines.
(C) It has interior points, namely it generates the whole vector space $\mathcal{T}$.

(D) It is closed.

(E) It is invariant with respect to the proper orthochronous Lorentz group which acts on the coordinates b^k and b^{rs} by means of its four-vector and antisymmetric tensor representations.

A simplified proof of this theorem is given in Appendix B.

The simplest description of this cone is given in terms of Dirac matrices. We adopt the metric tensor $g_{00} = -1$, $g_{11} = g_{22} = g_{33} = 1$, and we indicate by γ_k $(k = 0, 1, 2, 3)$ the real Dirac matrices in the Majorana representation. As is well known, there is a matrix C with the property

$$\gamma_k^T = -C^{-1}\gamma_k C \tag{3.1}$$

where the superscript T indicates the transposed matrix. In the representation we are using, we can put $C = \gamma_0$ and C is real antisymmetric. Then we represent the element (1.1) of $\mathcal{T}$ by means of the 4-dimensional real, symmetric matrix

$$\hat{b} = b^k C^{-1}\gamma_k - \tfrac{1}{2} b^{rs} C^{-1}\gamma_r\gamma_s \tag{3.2}$$

All the real, symmetric 4×4 matrices can be written in this form. The positive-semidefinite matrices are the coordinates of the elements of the cone $\mathcal{T}^+$. They are characterized by the property

$$u^T\hat{b}u \geq 0 \tag{3.3}$$

which must hold for every real spinor u. This cone is stable with respect to the transformations

$$\hat{b} \to (a^{-1})^T\hat{b}a^{-1} \tag{3.4}$$

where a is an arbitrary, real, nonsingular matrix, namely an element of the group $GL(4, R)$. We obtain in this way all the linear transformations of $\mathcal{T}$ which map the cone $\mathcal{T}^+$ onto itself.

It is clear that on the boundary of $\mathcal{T}^+$ we have $\det \hat{b} = 0$. A direct calculation, using an explicit representation of the Dirac matrices, shows that

$$\det \hat{b} = [(b^0)^2 - u]^2 - 8wb^0 - 4v \tag{3.5}$$

where

$$\begin{aligned} u &= \mathbf{b}^2 + \mathbf{b}'^2 + \mathbf{b}''^2 \\ v &= (\mathbf{b}\times\mathbf{b}')^2 + (\mathbf{b}'\times\mathbf{b}'')^2 + (\mathbf{b}''\times\mathbf{b})^2 \\ w &= \mathbf{b}''\cdot\mathbf{b}'\times\mathbf{b} \end{aligned} \tag{3.6}$$

The solutions of the equation

$$(t^2 - u)^2 - 8wt - 4v = 0 \tag{3.7}$$

are the eigenvalues of a real, symmetric matrix and therefore are real. We indicate them by $t_1 \leq t_2 \leq t_3 \leq t_4$. The cone $\mathcal{T}^+$ is defined by the inequality

$$b^0 \geq t_4 \tag{3.8}$$

In the absence of acceleration and rotation, we have $\mathbf{b}' = \mathbf{b}'' = 0$, $v = w = 0$ and from equations (3.7), (3.8) we get the inequality

$$b^0 \geq \sqrt{u} = \|\mathbf{b}\| \tag{3.9}$$

which is the usual relativistic upper bound to the velocity. If, as in the preceding section, we consider the moving rest frame of an extended particle, we have $\mathbf{v} = \mathbf{b} = 0$ and $w = 0$. Then equation (3.7) has the solution

$$t_4 = (u + 2\sqrt{v})^{1/2} = (\mathbf{b}'^2 + \mathbf{b}''^2 + 2\|\mathbf{b}' \times \mathbf{b}''\|)^{1/2} \tag{3.10}$$

and equation (3.8) coincides with equation (2.5) derived from the naive model.

4. CAUSAL INFLUENCE AND SPACETIME COINCIDENCE

We have seen that the cone $\mathcal{T}^+$ is invariant under the group $GL(4, R)$. It is natural to examine the possibility that all the fundamental laws of physics are symmetric with respect to this group or at least with respect to one of its subgroups $Sp(4, R)$ or $SL(4, R)$. This higher symmetry cannot be observed directly because the vacuum, described by the structure constants of the Poincaré group, is not invariant with respect to the whole symmetry group, but only with respect to a subgroup which coincides with the orthochronous Lorentz group.

The space $\mathcal{T}$ contains several sets stable under the action of $GL(4, R)$ besides the cone $\mathcal{T}^+$. They can be decomposed into orbits on which the group operates transitively. According to a classical result of the theory of the real quadratic forms, these orbits are characterized by the rank and the signature of the matrices $\hat{b}$ which label the points of $\mathcal{T}$. We indicate by $\mathcal{T}(m, n)$ the orbit defined by the matrices of rank $m+n$ with m positive and n negative eigenvalues. Since $m+n \leq 4$, there are 15 different orbits. The closure of an orbit is given by

$$\bar{\mathcal{T}}(m, n) = \bigcup_{\substack{p \leq m \\ q \leq n}} \mathcal{T}(p, q) \tag{4.1}$$

The cone $\mathcal{T}^+$ coincides with $\bar{\mathcal{T}}(4, 0)$. All these orbits can be described by means of inequalities similar to (3.8). For instance, the orbit $\mathcal{T}(1, 1)$ is given by

$$t_1 < t_2 = b^0 = t_3 < t_4 \tag{4.2}$$

The physical meaning of the orbits defined above can be clarified by studying some $SL(4, R)$ covariant quantum free fields on a flat space $\mathscr{S}$, namely on the affine space on which $\mathscr{T}$ acts as the translation group (Toller, 1988). It is assumed that a flat space $\mathscr{S}$ can approximate a very small region of the curved physical space $\mathscr{S}$. It turns out that if $\phi(s)$ is a scalar field, we have (in the sense of distribution theory)

$$[\phi(s), \phi(s')] = 0 \qquad \text{for} \quad s' - s \in \mathscr{T}(2, 2) \tag{4.3}$$

It is natural to assume that operations performed in frames separated by a vector belonging to $\mathscr{T}(2, 2)$ cannot influence each other, namely the two frames are causally independent. When we speak of an operation performed in the local frame s, we mean that this frame is used for positioning the experimental apparatus and that the apparatus acts in a very small neighborhood of the origin of the local frame s.

The complicated geometry of the space $\mathscr{T}$ suggests that there are two degrees of causal influence. If

$$s' - s \in \mathscr{T}^+ = \bar{\mathscr{T}}(4, 0) \tag{4.4}$$

we say that s' is strongly influenced by s and we write $s \Rightarrow s'$. When

$$s' - s \in \mathscr{T}_W = \bar{\mathscr{T}}(4, 0) \cup \bar{\mathscr{T}}(3, 1) \tag{4.5}$$

we say that s' is weakly influenced by s and we write $s \to s'$.

The two relations defined above have the following properties:

- (A) Both the strong and the weak causal influence are reflexive, namely we have $s \Rightarrow s$ and $s \to s$.
- (B) The strong causal influence is a transitive relation, namely $s \Rightarrow s'$ and $s' \Rightarrow s''$ imply $s \Rightarrow s''$.
- (C) The strong causal influence is an antisymmetric relation, namely $s \Rightarrow s'$ and $s' \Rightarrow s$ imply $s = s'$.
- (D) $s \Rightarrow s'$ and $s' \to s''$ imply $s \to s''$.
- (E) $s \to s'$ and $s' \Rightarrow s''$ imply $s \to s''$.

It follows from the properties A and D that the strong causal influence implies the weak causal influence. The property B follows from the convexity of $\mathscr{T}^+$. The weak causal influence is not transitive, because $\mathscr{T}_W$ is not convex. For instance, it may happen that

$$s'' - s' \in \mathscr{T}(3, 1), \qquad s' - s \in \mathscr{T}(3, 1), \qquad s'' - s \in \mathscr{T}(2, 2) \tag{4.6}$$

The properties D and E are consequences of the formula

$$\mathscr{T}_W + \mathscr{T}^+ = \mathscr{T}_W \tag{4.7}$$

which is evident if we remember that when we add a positive-semidefinite matrix to a Hermitian matrix, its eigenvalues do not decrease.

The property C follows from the fact that $\mathcal{T}^+$ does not contain straight lines. This is not true for weak causal influence, because we have

$$\mathcal{T}_W \cap (-\mathcal{T}_W) = \bar{\mathcal{T}}(3, 1) \cap \bar{\mathcal{T}}(1, 3) = \bar{\mathcal{T}}(1, 1) \tag{4.8}$$

and if $s' - s$ belongs to this set, s and s' can influence each other weakly. In this case we write $s \leftrightarrow s'$ and we say that the set (4.8) describes a reciprocal (weak) causal influence. Since the weak causal influence is not transitive, the reciprocal causal influence is not an equivalence relation.

In order to show that a structure defined by the properties A–E is rather natural, we suggest a model, completely different from the one considered above, which has all these properties. We start from a spacetime $\mathcal{M}$ provided with a reflexive antisymmetric transitive relation of causal influence and we consider a family $\mathcal{S}$ of sets of pairwise causally independent points of $\mathcal{M}$. If $s, s' \in \mathcal{S}$, we say that $s \Rightarrow s'$ if every element of s' is influenced by at least one element of s and every element of s influences at least one element of s'. We say also that $s \to s'$ if at least one element of s' is influenced by at least one element of s. We leave to the reader the proof of the properties A–E. This model suggests that the concept of weak causal influence is related in some way to the necessity of considering extended objects.

In order to extend these concepts to a curved space $\mathcal{S}$, one should start from an analysis of the corresponding quantum fields, which has been done only in very particular cases. Here we propose a definition of strong and weak causal influence which preserves at least some of the properties A–E that these relations have in a flat space $\mathcal{S}$. The main purpose of the following sections is to show that these definitions do not contradict the direct physical evidence.

The basic concept we shall use is the exponential mapping, which, under some conditions, defines a diffeomorphism of $\mathcal{S}$ starting from a vector field on $\mathcal{S}$. For instance, given an element $B = b^\alpha A_\alpha \in \mathcal{T}$ and a point $s \in \mathcal{S}$, one can define the point $s' = \exp(B)s \in \mathcal{S}$ by means of the differential equation

$$\frac{ds(\lambda)}{d\lambda} = b^\alpha A_\alpha(s(\lambda)), \qquad s(0) = s, \qquad s(1) = s' \tag{4.9}$$

In the following we assume that this equation has solutions defined for all the values of the variable λ.

The strong causal influence $s \Rightarrow s'$ cannot be characterized by the relation $s' \in \exp(\mathcal{T}^+)s$, which in general does not define a transitive relation. The correct statement is that $s \Rightarrow s'$ if and only if there is a curve $\lambda \to s(\lambda)$ with the properties

$$\frac{ds(\lambda)}{d\lambda} \in \mathcal{T}^+, \qquad s(0) = s, \qquad s(1) = s' \tag{4.10}$$

This relation is clearly transitive and reflexive. It is also antisymmetric if we assume that all the closed curves with the property (4.10) are reduced to a single point.

The simplest assumption concerning weak causal influence is that $s \to s'$ if and only if

$$s' \in \exp(\mathcal{T}_W)s \tag{4.11}$$

Unfortunately, as we shall see in Section 6, the properties D and E are not satisfied in the physically most relevant case. We shall adopt this assumption in the absence of a more satisfactory one. If

$$s' \in \exp[\bar{\mathcal{T}}(1, 1)]s \tag{4.12}$$

we have reciprocal causal influence between s and s'. However, since the exponential mapping may be noninjective, one cannot prove in general that this condition is necessary.

If $\mathcal{T}$ is the Lie algebra of a Lie group $\mathcal{P}$, $\exp(B)$ can be considered as an element of $\mathcal{P}$ which acts on $\mathcal{S}$. The definition (4.10) of strong causal influence means that $s' \in \mathcal{P}^+ s$, where $\mathcal{P}^+ \subset \mathcal{P}$ is the closed semigroup generated by the set $\exp(\mathcal{T}^+)$ (which in general is not a semigroup). The transitivity follows from the fact that $\mathcal{P}^+$ is a semigroup. The weak causal influence is described by the set $\exp(\mathcal{T}_W) \subset \mathcal{P}$, which is not a semigroup.

In special relativity, without any limitation to acceleration and angular velocity, there is only one kind of causal influence defined by the relation

$$x'^0 - x^0 \geq \|\mathbf{x}' - \mathbf{x}\| \tag{4.13}$$

between the coordinates in Minkowski spacetime. This relation is transitive and it is antisymmetric in spacetime. However, it is not antisymmetric in the bundle $\mathcal{S}$ of the reference frames. Frames which differ by a homogeneous Lorentz transformation have a reciprocal causal influence and their origins coincide in spacetime. In the usual relativistic theories the reciprocal causal influence is an equivalence relation and it is the same as the spacetime coincidence. This is true also in general relativity, which, as stressed by Einstein himself, is based on the possibility of defining the spacetime coincidence of two events.

In the theories we are considering, spacetime coincidence is not a concept invariant under $GL(4, R)$ (as time coincidence or contemporaneity is not a Lorentz-invariant concept). It has to be replaced by the concept of reciprocal causal influence, which, however, is not an equivalence relation. In this sense, these theories are not local. We shall see that this nonlocality is far from being observable with available experimental methods.

5. DETAILED DESCRIPTION OF SOME ORBITS IN $\mathscr{T}$

In order to find explicit solutions of (3.7), it is convenient to perform a homogeneous Lorentz transformation in such a way that the antisymmetric tensor b^{rs} takes a simple form, as discussed, for instance, in the book by Landau and Lifshitz (1971) in the case of the electromagnetic field tensor. In the following, we consider only orthochronous Lorentz transformations. One has to distinguish a generic case and a degenerate case. In the degenerate case, both the Lorentz-invariant quantities

$$\tfrac{1}{8}\varepsilon_{rsuv}b^{rs}b^{uv} = \mathbf{b}' \cdot \mathbf{b}'', \qquad \tfrac{1}{2}b^{rs}b_{rs} = \mathbf{b}'^2 - \mathbf{b}''^2 \tag{5.1}$$

vanish and by means of a suitable rotation we get

$$\mathbf{b}' = (\alpha, 0, 0), \qquad \mathbf{b}'' = (0, \alpha, 0), \qquad \alpha \geq 0 \tag{5.2}$$

If we put

$$b^0 = t, \qquad \mathbf{b} = (\rho \cos\phi, \rho \sin\phi, \gamma), \qquad \rho \geq 0 \tag{5.3}$$

equation (3.7) after some calculations takes the form

$$(t^2 - \gamma^2 - \rho^2)^2 - 4\alpha^2(t-\gamma)^2 = 0 \tag{5.4}$$

This equation can be solved immediately and we get

$$\begin{aligned} t_1 &= -\alpha - [(\alpha+\gamma)^2 + \rho^2]^{1/2} \\ t_2 &= \alpha - [(\alpha-\gamma)^2 + \rho^2]^{1/2} \\ t_3 &= -\alpha + [(\alpha+\gamma)^2 + \rho^2]^{1/2} \\ t_4 &= \alpha + [(\alpha-\gamma)^2 + \rho^2]^{1/2} \end{aligned} \tag{5.5}$$

In the generic case at least one of the expressions (5.1) is nonvanishing and after a Lorentz transformation one can put

$$\mathbf{b}' = (0, 0, \alpha), \qquad \mathbf{b}'' = (0, 0, \beta), \qquad \alpha^2 + \beta^2 > 0 \tag{5.6}$$

Using again equation (5.3), we can write equation (3.7) in the simple form

$$(t^2 - \alpha^2 - \beta^2 - \gamma^2 - \rho^2)^2 - 4\rho^2(\alpha^2+\beta^2) = 0 \tag{5.7}$$

and its solutions are given by

$$\begin{aligned} t_1 &= -\{[\rho + (\alpha^2+\beta^2)^{1/2}]^2 + \gamma^2\}^{1/2} \\ t_2 &= -\{[\rho - (\alpha^2+\beta^2)^{1/2}]^2 + \gamma^2\}^{1/2} \\ t_3 &= \{[\rho - (\alpha^2+\beta^2)^{1/2}]^2 + \gamma^2\}^{1/2} \\ t_4 &= \{[\rho + (\alpha^2+\beta^2)^{1/2}]^2 + \gamma^2\}^{1/2} \end{aligned} \tag{5.8}$$

The Lorentz transformation which permits us to write the vectors $\mathbf{b}'$ and $\mathbf{b}''$ in the simplified form (5.6) can be decomposed into a rotation, which gives

$$\begin{aligned} \mathbf{b}' &= (0, \beta \sinh \zeta, \alpha \cosh \zeta) \\ \mathbf{b}'' &= (0, -\alpha \sinh \zeta, \beta \cosh \zeta), \qquad \zeta \geq 0 \end{aligned} \tag{5.9}$$

$$\begin{aligned} b^0 &= t \cosh \zeta + \rho \cos \phi \sinh \zeta \\ \mathbf{b} &= (\rho \cos \phi \cosh \zeta + t \sinh \zeta, \rho \sin \phi, \gamma) \end{aligned} \tag{5.10}$$

followed by a Lorentz boost along the coordinate b^1, which leads to equations (5.6) and (5.3). From equations (5.9) and (5.10) we see that it is always possible to express α, β, and ζ in terms of the rotational invariants and we get

$$\begin{aligned} \alpha^2 - \beta^2 &= \mathbf{b}'^2 - \mathbf{b}''^2, \qquad \alpha\beta = \mathbf{b}' \cdot \mathbf{b}'' \\ \alpha^2 + \beta^2 &= [(\mathbf{b}'^2 - \mathbf{b}''^2)^2 + 4(\mathbf{b}' \cdot \mathbf{b}'')^2]^{1/2} \end{aligned} \tag{5.11}$$

$$\mathbf{b}'^2 + \mathbf{b}''^2 = (\alpha^2 + \beta^2) \cosh(2\zeta) \tag{5.12}$$

Note that

$$|\mathbf{b}'^2 - \mathbf{b}''^2| \leq \alpha^2 + \beta^2 \leq \mathbf{b}'^2 + \mathbf{b}''^2 \tag{5.13}$$

The set $\bar{\mathscr{T}}(1, 0) = -\bar{\mathscr{T}}(0, 1)$ is defined by the condition $t = t_2 = t_3 = t_4$. Since these equalities are not compatible with equations (5.8), we have to consider only the degenerate case and from equations (5.5) we obtain

$$t = \gamma = \alpha, \qquad \rho = 0 \tag{5.14}$$

If we substitute these equalities into equations (5.2) and (5.3) and we perform an arbitrary rotation, we see that the set $\bar{\mathscr{T}}(1, 0)$ is defined by the following equations:

$$b^0 = \|\mathbf{b}\| = \|\mathbf{b}'\| = \|\mathbf{b}''\|, \qquad \mathbf{b}' \cdot \mathbf{b}'' = 0, \qquad b^0\mathbf{b} = \mathbf{b}' \times \mathbf{b}'' \tag{5.15}$$

These conditions mean that if not all the components b^α vanish, we have $b^0 > 0$, and $\mathbf{b}/b^0, \mathbf{b}'/b^0, \mathbf{b}''/b^0$ form a left-handed triad of orthogonal unit vectors in three-dimensional Euclidean space.

The set $\bar{\mathscr{T}}(1, 1)$ is defined by the condition $t = t_2 = t_3$ and in the degenerate case from equations (5.5) we have

$$\begin{aligned} 0 &= [(\alpha + \gamma)^2 + \rho^2]^{1/2} + [(\alpha - \gamma)^2 + \rho^2]^{1/2} - 2\alpha \\ &\geq |\alpha + \gamma| + |\alpha - \gamma| - 2\alpha \geq 0 \end{aligned} \tag{5.16}$$

It follows that

$$\rho = 0, \qquad t = \gamma, \qquad |\gamma| \leq \alpha \tag{5.17}$$

In the generic case from equations (5.8) we obtain

$$t=0, \qquad \gamma=0, \qquad \rho=(\alpha^2+\beta^2)^{1/2} \tag{5.18}$$

This equation shows that if we fix the vectors $\mathbf{b}'$ and $\mathbf{b}''$, the points of $\bar{\mathcal{T}}(1,1)$ form an ellipse in a spacelike plane contained in the space of the variables b^k. In the degenerate case the ellipse becomes a lightlike segment described by equations (5.17). We see in this way that $\bar{\mathcal{T}}(1,1)$ is a 7-dimensional surface.

If we substitute equations (5.18) into equations (5.10) and use equation (5.12), we can derive the following formulas:

$$b^0=[\tfrac{1}{2}(\mathbf{b}'^2+\mathbf{b}''^2-\alpha^2-\beta^2)]^{1/2}\cos\phi \tag{5.19}$$

$$\|\mathbf{b}\|=[(b^0)^2+\alpha^2+\beta^2]^{1/2} \tag{5.20}$$

From these formulas and equation (5.13) we get the following rotation-invariant inequalities, which are satisfied also in the degenerate case and hold on the whole set $\bar{\mathcal{T}}(1,1)$:

$$|b^0|\le\min(\|\mathbf{b}'\|,\|\mathbf{b}''\|), \qquad \|\mathbf{b}\|\le(\mathbf{b}'^2+\mathbf{b}''^2)^{1/2} \tag{5.21}$$

The set $\mathcal{T}^+=\bar{\mathcal{T}}(4,0)=-\bar{\mathcal{T}}(0,4)$ is defined by the condition $t\ge t_4$. In the generic case from equations (5.8) we have

$$(\rho^2+\gamma^2+\alpha^2+\beta^2)^{1/2}\le t_4\le(\rho^2+\gamma^2)^{1/2}+(\alpha^2+\beta^2)^{1/2} \tag{5.22}$$

A first consequence is given by the Lorentz-invariant inequality [see also equation (B.4) of Appendix B]:

$$b^0\ge\{\mathbf{b}^2+[(\mathbf{b}'^2-\mathbf{b}''^2)^2+4(\mathbf{b}'\cdot\mathbf{b}'')^2]^{1/2}\}^{1/2}\ge\|\mathbf{b}\| \tag{5.23}$$

which can be extended by continuity to the degenerate case and holds for all the points of $\mathcal{T}^+$. We see that the cone $\mathcal{T}^+$ imposes a limitation on the velocity which may be stronger than the usual one if the frame is accelerated or rotated.

The set $\bar{\mathcal{T}}(3,1)=-\bar{\mathcal{T}}(1,3)$ is defined by the condition $t_3\le t\le t_4$. In the generic case from equations (5.8) we get

$$|(\rho^2+\gamma^2)^{1/2}-(\alpha^2+\beta^2)^{1/2}|\le t_3\le t\le t_4\le(\rho^2+\gamma^2)^{1/2}+(\alpha^2+\beta^2)^{1/2} \tag{5.24}$$

In particular, $t\ge 0$ and after an arbitrary Lorentz transformation we get the inequality

$$b^0\ge-\|\mathbf{b}\| \tag{5.25}$$

which holds on the whole set $\mathcal{T}_W$.

If x is a 4-vector with $x^0\ne\|\mathbf{x}\|$ and $y=\Lambda x$, where Λ is a Lorentz transformation with rapidity $\zeta\ge 0$, one can show that

$$\exp(-\zeta)\le\frac{y^0-\|\mathbf{y}\|}{x^0-\|\mathbf{x}\|}\le\exp\zeta, \qquad \cosh\zeta=\Lambda^0_0 \tag{5.26}$$

From this general inequality and from equations (5.12) and (5.24), we get

$$|b^0-\|\mathbf{b}\||\leq|t-(\rho^2+\gamma^2)^{1/2}|\exp\zeta\leq(\alpha^2+\beta^2)^{1/2}\exp\zeta$$
$$=[\tfrac{1}{2}(\mathbf{b}'^2+\mathbf{b}''^2+\alpha^2+\beta^2)]^{1/2}+[\tfrac{1}{2}(\mathbf{b}'^2+\mathbf{b}''^2-\alpha^2-\beta^2)]^{1/2} \quad (5.27)$$

This inequality can be extended by continuity to the degenerate case and it holds on the whole set $\bar{\mathcal{T}}(3,1)$. By means of a further majorization we get the weaker, but simpler inequality

$$|b^0-\|\mathbf{b}\||\leq[2(\mathbf{b}'^2+\mathbf{b}''^2)]^{1/2} \quad (5.28)$$

As a consequence, on the set $\mathcal{T}_W$ we have

$$b^0\geq\|\mathbf{b}\|-[2(\mathbf{b}'^2+\mathbf{b}''^2)]^{1/2} \quad (5.29)$$

and

$$b^0\geq\|\mathbf{b}\|+[2(\mathbf{b}'^2+\mathbf{b}''^2)]^{1/2} \quad (5.30)$$

is a sufficient condition for belonging to $\mathcal{T}^+$.

These inequalities show that, if we can disregard lengths and time intervals of the order of $l(\mathbf{b}'^2+\mathbf{b}''^2)^{1/2}$, both the strong and the weak causal influence can be described by the usual inequality $b^0\geq\|\mathbf{b}\|$. Equation (5.21) shows that, under the same conditions, the reciprocal causal influence cannot be distinguished from the spacetime coincidence. In the next section we shall see how these conclusions are modified by the curvature of the space $\mathcal{S}$.

6. CAUSAL INFLUENCE IN THE POINCARÉ GROUP

In this section we study some relevant properties of the sets $\mathcal{P}^+$ and $\exp(\mathcal{T}_W)$ when $\mathcal{P}$ is the proper orthochronous Poincaré group. According to our assumptions, these sets describe strong and weak causal influence in the physically relevant case in which $\mathcal{S}$ is the bundle of frames of the Minkowski spacetime. For our purposes, it is convenient to indicate by $T(x)=T(x^k)=\exp(x^kA_k)$ an element of the subgroup of translations. Then every element of $\mathcal{P}$ can be written in the form $g=\Lambda T(x)$, where Λ is an element of the homogeneous Lorentz subgroup and x^k are the coordinates of the origin of the frame $s'=gs$ measured in the frame s. If we indicate by Λ both the 4×4 matrix which acts on the 4-vectors and the linear operator of the adjoint representation which acts on the Lie algebra $\mathcal{T}$, we can write

$$\Lambda\exp(B)\Lambda^{-1}=\exp(\Lambda B) \quad (6.1)$$

and in particular

$$\Lambda T(x)\Lambda^{-1}=T(\Lambda x) \quad (6.2)$$

From the Lorentz symmetry of the sets $\mathcal{T}^+$ and $\mathcal{T}_W$, we get the following relations:

$$\Lambda\mathcal{P}^+\Lambda^{-1}=\mathcal{P}^+, \qquad \Lambda\exp(\mathcal{T}_W)\Lambda^{-1}=\exp(\mathcal{T}_W) \tag{6.3}$$

The same symmetry permits us to write the definition (4.10) of strong causal influence in a more convenient way. In the special case we are considering, it gives rise to the following definition: $g\in\mathcal{P}^+$ if and only if there is a curve $\lambda\to g(\lambda)\in\mathcal{P}$ which, disregarding terms of the second order in ε, has the properties

$$\begin{gathered} g(\lambda+\varepsilon)=\exp[\varepsilon B(\lambda)]g(\lambda) \\ B(\lambda)\in\mathcal{T}^+, \qquad g(0)=1, \qquad g(1)=g \end{gathered} \tag{6.4}$$

If we write $g(\lambda)=\Lambda(\lambda)T(x(\lambda))$ and we take into account the Lorentz invariance of $\mathcal{T}^+$, we can write equation (6.4) in the form

$$\Lambda(\lambda+\varepsilon)T(x(\lambda+\varepsilon))=\Lambda(\lambda)\exp[\varepsilon B'(\lambda)]T(x(\lambda)) \tag{6.5}$$

where

$$B'(\lambda)=\Lambda^{-1}(\lambda)B(\lambda)=b^k(\lambda)A_k+\tfrac{1}{2}b^{rs}(\lambda)A_{rs}\in\mathcal{T}^+ \tag{6.6}$$

It follows that, always disregarding higher order terms in ε,

$$\Lambda(\lambda+\varepsilon)=\Lambda(\lambda)\exp[\tfrac{1}{2}\varepsilon b^{rs}(\lambda)A_{rs}], \qquad \Lambda(0)=1, \qquad \Lambda(1)=\Lambda \tag{6.7}$$

$$x^k(\lambda+\varepsilon)=\varepsilon b^k(\lambda)+x^k(\lambda), \qquad x^k(0)=0, \qquad x^k(1)=x^k \tag{6.8}$$

Equations (6.6)-(6.8) are equivalent to the definition (6.4) of $\mathcal{P}^+$, but they have the advantage that the translations and the Lorentz transformations appear in different equations.

It follows from equations (6.6) and (5.23) that $b^0(\lambda)\geq\|\mathbf{b}(\lambda)\|$, and from equations (6.8) we obtain

$$0\leq\|\mathbf{x}\|\leq\int_0^1\|\mathbf{b}(\lambda)\|\,d\lambda\leq\int_0^1 b^0(\lambda)\,d\lambda=x^0 \tag{6.9}$$

We see that $\Lambda T(x)\in\mathcal{P}^+$ implies $x^0\geq\|\mathbf{x}\|$. It follows also that a special kind of element of $\mathcal{P}^+$ is given by

$$g=\exp(\tfrac{1}{2}b^{rs}A_{rs})T(b^k), \qquad b^kA_k+\tfrac{1}{2}b^{rs}A_{rs}\in\mathcal{T}^+ \tag{6.10}$$

In particular, from (5.22) we see that $\mathcal{P}^+$ contains all the elements of the form

$$g=\exp(\alpha A_{12}+\beta A_{30})T(x), \qquad x^0\geq\|\mathbf{x}\|+(\alpha^2+\beta^2)^{1/2} \tag{6.11}$$

and from rotational invariance we see that it contains elements of the kind

$$g=RT(x), \qquad x^0\geq\|\mathbf{x}\|+\phi \tag{6.12}$$

where R is a rotation of an angle $0\leq\phi\leq\pi$ and

$$g=LT(x), \qquad x^0\geq\|\mathbf{x}\|+\eta \tag{6.13}$$

where L is a pure Lorentz transformation (boost) with rapidity $\eta\geq0$.

Since every proper orthochronous Lorentz transformation can be written in the form

$$\Lambda = R_1 L R_2 = R_1 (L R_2 R_1) R_1^{-1} \tag{6.14}$$

by using also the rotation invariance, we have the following sufficient condition for $\Lambda T(x) \in \mathscr{P}^+$:

$$x^0 \geq \|\mathbf{x}\| + \eta(\Lambda) + \pi \tag{6.15}$$

where $\eta(\Lambda)$ is the rapidity of the Lorentz transformation Λ. From equation (6.3) we see that the same condition ensures also that $T(x)\Lambda \in \mathscr{P}^+$.

In order to treat the set $\exp(\mathscr{T}_W)$, we have to consider the exponential mapping in the Poincaré group. If $B \in \mathscr{T}$ is given by equation (1.1), one can always write

$$\exp(\lambda B) = \exp(\tfrac{1}{2}\lambda b^{rs} A_{rs}) T(x^k(\lambda)) \tag{6.16}$$

Disregarding terms of the order ε^2, we have

$$\begin{aligned} &\exp[\tfrac{1}{2}(\lambda + \varepsilon) b^{rs} A_{rs}] T(x^k(\lambda + \varepsilon)) \\ &\quad = \exp((\lambda + \varepsilon) B) \\ &\quad = \exp(\tfrac{1}{2}\lambda b^{rs} A_{rs}) T(x^k(\lambda)) \exp(\tfrac{1}{2}\varepsilon b^{rs} A_{rs}) T(\varepsilon b^k) \end{aligned} \tag{6.17}$$

and therefore

$$\begin{aligned} T(x^k(\lambda + \varepsilon) - \varepsilon b^k) &= \exp(-\tfrac{1}{2}\varepsilon b^{rs} A_{rs}) T(x^k(\lambda)) \exp(\tfrac{1}{2}\varepsilon b^{rs} A_{rs}) \\ &= T(x^k(\lambda) + \varepsilon b^{ik} x_i(\lambda)) \end{aligned} \tag{6.18}$$

This formula is equivalent to the differential equation

$$\frac{dx^k(\lambda)}{d\lambda} = b^k + b^{ik} x_i(\lambda), \qquad x^k(0) = 0, \qquad x^k(1) = x^k \tag{6.19}$$

In order to solve this equation, it is convenient to simplify the tensor b^{rs} by means of a Lorentz transformation, as in the preceding section. It is sufficient to consider the generic case, because our results can be extended to the degenerate case by continuity. From (5.3) and (5.6) we have

$$\begin{aligned} \frac{dx^0}{d\lambda} &= t + \beta x^3 \\ \frac{dx^1}{d\lambda} &= \rho \cos\phi - \alpha x^2 \\ \frac{dx^2}{d\lambda} &= \rho \sin\phi + \alpha x^1 \\ \frac{dx^3}{d\lambda} &= \gamma + \beta x^0 \end{aligned} \tag{6.20}$$

and the solution is

$$\begin{aligned} x^0 &= \gamma\beta^{-1}(\cosh\beta - 1) + t\beta^{-1}\sinh\beta \\ x^1 &= 2\rho\alpha^{-1}\sin(\tfrac{1}{2}\alpha)\cos(\phi + \tfrac{1}{2}\alpha) \\ x^2 &= 2\rho\alpha^{-1}\sin(\tfrac{1}{2}\alpha)\sin(\phi + \tfrac{1}{2}\alpha) \\ x^3 &= t\beta^{-1}(\cosh\beta - 1) + \gamma\beta^{-1}\sinh\beta \end{aligned} \tag{6.21}$$

It follows that

$$\|\mathbf{x}\| = \left[(x^0)^2 - (t^2 - \gamma^2)\left(\frac{2}{\beta}\sinh\frac{\beta}{2}\right)^2 + \rho^2\left(\frac{2}{\alpha}\sin\frac{\alpha}{2}\right)^2\right]^{1/2} \tag{6.22}$$

Now we use the general inequality

$$[A^2 + (B - C)^2]^{1/2} - (A^2 + B^2)^{1/2} + C \geq 0, \qquad C \geq 0 \tag{6.23}$$

which follows from the remark that the derivative of the left-hand side with respect to C is nonnegative. We obtain

$$\|\mathbf{x}\| \leq [(x^0)^2 + X]^{1/2} + (\alpha^2 + \beta^2)^{1/2}\left|\frac{2}{\alpha}\sin\frac{\alpha}{2}\right| \tag{6.24}$$

where

$$X = [\rho - (\alpha^2 + \beta^2)^{1/2}]^2\left(\frac{2}{\alpha}\sin\frac{\alpha}{2}\right)^2 - (t^2 - \gamma^2)\left(\frac{2}{\beta}\sinh\frac{\beta}{2}\right)^2 \tag{6.25}$$

In the set $\mathscr{T}_W$, we have $t \geq t_3$ and from equation (5.8)

$$t^2 - \gamma^2 \geq [\rho - (\alpha^2 + \beta^2)^{1/2}]^2, \qquad t \geq |\gamma| \tag{6.26}$$

Moreover,

$$\left(\frac{2}{\alpha}\sin\frac{\alpha}{2}\right)^2 \leq 1, \qquad \left(\frac{2}{\beta}\sinh\frac{\beta}{2}\right)^2 \geq 1 \tag{6.27}$$

and it follows that $X \leq 0$. In conclusion, from equations (6.21) and (6.24), if b^{rs} is given by equation (5.6), on the set $\exp(\mathscr{T}_W)$ we have

$$x^0 \geq 0, \qquad x^0 - \|\mathbf{x}\| \geq -(\alpha^2 + \beta^2)^{1/2}\left|\frac{2}{\alpha}\sin\frac{\alpha}{2}\right| \tag{6.28}$$

After a Lorentz transformation we obtain the inequality

$$x^0 \geq -\|\mathbf{x}\| \tag{6.29}$$

which holds on the whole set $\exp(\mathcal{T}_W)$. If we use equation (5.26), from equations (6.28) we get

$$x^0 - \|\mathbf{x}\| \geq -(\alpha^2+\beta^2)^{1/2} \left|\frac{2}{\alpha}\sin\frac{\alpha}{2}\right| \exp\zeta \tag{6.30}$$

In this case the homogeneous Lorentz transformation

$$\Lambda = \exp(\tfrac{1}{2}b^{rs}A_{rs}) = \exp(-\zeta A_{10})\exp(\alpha A_{12}+\beta A_{30})\exp(\zeta A_{10}) \tag{6.31}$$

has a rapidity $\eta(\Lambda) \geq 0$ given by

$$\cosh\eta(\Lambda) = \Lambda^0_0 = (\cosh\zeta)^2\cosh\beta - (\sinh\zeta)^2\cos\alpha \geq \cosh\beta \tag{6.32}$$

We use this formula to substitute $\exp\zeta$ in equation (6.30) and we obtain a long expression, which is minimal for $\beta = 0$, $\cos\alpha = -1$. In this way we get the following inequality valid for $\Lambda T(x) \in \exp(\mathcal{T}_W)$:

$$x^0 - \|\mathbf{x}\| \geq -2\exp[\tfrac{1}{2}\eta(\Lambda)] \tag{6.33}$$

Note that in this formula the equality sign holds, for instance, if we take an element of $\mathcal{T}_W$ given by (5.9), (5.10) with

$$\alpha = \pi, \qquad \beta = 0, \qquad \rho = \pi, \qquad \gamma = 0, \qquad \phi = \tfrac{1}{2}\pi, \qquad t = 0 \tag{6.34}$$

In fact, in this case we have

$$\begin{gathered}\Lambda = \exp(\pi\cosh\zeta A_{12} - \pi\sinh\zeta A_{20}) \\ \eta(\Lambda) = 2\zeta, \qquad x = (-2\sinh\zeta, -2\cosh\zeta, 0, 0)\end{gathered} \tag{6.35}$$

Two frames have a reciprocal causal influence if they are connected by a transformation belonging to $\exp(\mathcal{T}_W) \cap \exp(-\mathcal{T}_W)$. In this case we must have

$$|x^0| + \|\mathbf{x}\| \leq 2\exp\frac{\eta}{2} \tag{6.36}$$

Finally, we consider the set $\mathcal{P}^+\exp(\mathcal{T}_W)$, which would coincide with $\exp(\mathcal{T}_W)$ if the property E of Section 4 were valid. It is sufficient to characterize the set $\mathcal{U}$ which contains the four-vectors x with the property

$$T(x) \in \mathcal{P}^+\exp(\mathcal{T}_W) \tag{6.37}$$

From equations (6.15) we see that

$$T(y)\Lambda^{-1}\Lambda T(x) = T(x+y) \in \mathcal{U} \tag{6.38}$$

where Λ and x are given by equation (6.35) and $y = (\eta(\Lambda)+\pi, 0, 0, 0)$. If we take η sufficiently large, we see that $\mathcal{U}$ contains spacelike vectors. We remark that $\mathcal{U}$ is Lorentz invariant and has the property $x+y \in \mathcal{U}$ if $x \in \mathcal{U}$

and $y^0 \geq \|\mathbf{y}\|$. It follows that it contains all the vectors with the property $x^0 > -\|\mathbf{x}\|$. However, we have $T(x) \in \exp(\mathcal{T}_W)$ only if $x^0 \geq \|\mathbf{x}\|$ and the property E of Section 4 is not satisfied. In a similar way one can show that the property D is not satisfied.

7. CONCLUSIONS

In the preceding sections we have examined the consequences of a limitation on acceleration and angular velocity described by a cone $\mathcal{T}^+$ in the Lie algebra $\mathcal{T}$ of the Poincaré group or, for more general situations, in the tangent spaces of the 10-dimensional manifold $\mathcal{S}$ of the local reference frames. According to Brandt (1983, 1987), these limitations are expected effects of quantum gravity. In the absence of a consistent quantum theory of the gravitational field, the form of the cone $\mathcal{T}^+$ has been determined starting from rather reasonable general assumptions.

The equation (3.7) satisfied by the boundary of $\mathcal{T}^+$ defines a larger surface which divides the space $\mathcal{T}$ into five regions, which we have interpreted in terms of the relations of causal independence and of strong, weak, and reciprocal causal influence. These relations are defined in a small region of $\mathcal{S}$, where it can be confused with the tangent space, but we have suggested a natural extension of these relations to distant points of $\mathcal{S}$. In Section 6 we have studied in some detail the case in which $\mathcal{S}$ is the bundle of frames of the Minkowski spacetime, diffeomorphic to the Poincaré group.

These calculations suggest that the spacetime coincidence of two frames moving with relative velocity $tgh\,\eta$ has an indeterminacy $\Delta x = l \exp(\frac{1}{2}\eta)$, where l is of the order of the Planck length, namely $l \approx 10^{-35}$ m. The inequalities (6.9), (6.15), (6.33), and (6.36) show that, if we can disregard lengths and time intervals of the order of this quantity, the relations of causal influence we are considering cannot be distinguished from the usual causal relations of special relativity. Also, the reciprocal causal influence cannot be distinguished from the usual spacetime coincidence.

We remark that the largest velocity appearing in experiments concerns electrons in high-energy accelerators and we have $\exp \eta \approx 2P/M < 10^6$. As a consequence, we might have nonlocal effects at distances of at most $l \cdot 10^3 \approx 10^{-32}$ m. The smallest distance one can explore is the wavelength corresponding to a momentum transfer in scattering experiments and is larger than 10^{-18} m. It follows that the deviations from the usual locality and causality assumptions cannot be observed directly. This conclusion holds for all the theories based on the Planck length.

The deviation mentioned above could have indirect consequences on the structure of elementary particles and, as suggested by Gasperini (1987), on the very early cosmology. In this respect, it could be interesting to analyze the causal relations in the bundle of frames of a spacetime endowed with a Robertson-Walker metric.

APPENDIX A: MINIMAL RADIUS OF AN OBJECT

In this Appendix we show that quantum gravitational effects give rise to a lower limit to the observable radius of a material body. Since we do not have a consistent quantum theory of gravitation, we have to use the concepts of quantum theory and of general relativity in a complementary way, analogous to the complementary treatment of the wave and particle models suggested by Bohr when the mathematical formalism of quantum theory was not yet available. We shall disregard systematically numerical factors of the order of unity.

We consider an experiment which establishes that an object (not necessarily a single particle) is contained in a sphere of radius R during a time interval T. According to quantum theory, during the measurement the energy of the object can change by an undetermined amount of the order of hT^{-1}. As a consequence, there is a large probability that the mass M of the object satisfies the inequality

$$Mc^2 \geq hT^{-1} \tag{A.1}$$

From quantum theory it also follows that the object has an undetermined momentum and an undetermined velocity of the order of $hR^{-1}M^{-1}$. As a consequence, during the time T its center of mass has an undetermined displacement (spreading of the wave packet) and if we require that it stays in a sphere of radius R, we obtain the inequality

$$R \geq hR^{-1}M^{-1}T \tag{A.2}$$

From general relativity, we know that a body with mass M cannot be smaller than the Schwarzschild radius. It follows that

$$R \geq c^{-2}GM \tag{A.3}$$

We can eliminate M from (A.1)-(A.3) and we get

$$RT \geq hGc^{-4} = l_P^2 c^{-1} \tag{A.4}$$

$$R^3 \geq hc^{-2}GT = l_P^2 cT \tag{A.5}$$

In particular, from the last two equations we obtain

$$R \geq l_P \tag{A.6}$$

APPENDIX B: CHARACTERIZATION OF THE CONE $\mathscr{T}^+$

In this Appendix we simplify the characterization of the cone $\mathscr{T}^+$ by proving the following result.

Proposition. If a cone $\mathscr{T}^+ \subset \mathscr{T}$ satisfies the conditions A–E of Section 3, it is given, after a possible change of sign of the coordinates b^k, by the closure of the set composed of the sums of pairs of elements with the properties

$$\|\mathbf{b}\| = b^0, \qquad \|\mathbf{b}'\| = \|\mathbf{b}''\| = pb^0, \qquad \mathbf{b}' \cdot \mathbf{b}'' = 0, \qquad p^2 b^0 \mathbf{b} = \mathbf{b}' \times \mathbf{b}'' \tag{B.1}$$

$$0 \leq p \leq l^{-1} \tag{B.2}$$

where l is a given positive constant.

If $p > 0$, the last two conditions of (B.1) mean that $\mathbf{b}$, $\mathbf{b}'$, and $\mathbf{b}''$ form a left-handed triad of orthogonal vectors. We can put $l = 1$ by means of a rescaling of the coordinates b^k, which can be interpreted as a suitable choice of the units of length and time (we have already chosen $c = 1$). Actually, all the elements of $\mathscr{T}^+$ are the sum of two elements with the properties (B.1), (B.2), but this stronger result is not necessary for the characterization of $\mathscr{T}^+$. When the uniqueness of $\mathscr{T}^+$ has been proven, all its properties are more easily derived from its representation in terms of 4×4 matrices.

First of all, we remark that if a convex rotation-invariant set in $\mathscr{T}$ contains an element with coordinates $(b^0, \mathbf{b}, \mathbf{b}', \mathbf{b}'')$, it contains also the element with coordinates $(b^0, 0, 0, 0)$ which belongs to the convex hull of a finite set of points obtained from the given point by means of suitable rotations. From the condition C, we see that $\mathscr{T}^+$ has points with $b^0 \neq 0$ and after a possible change of sign of the coordinates b^k we can assume that it has points with $b^0 > 0$. From the conditions A, B, and D it follows that $\mathscr{T}^+$ contains all the points with the properties

$$b^0 \geq 0, \qquad \mathbf{b} = \mathbf{b}' = \mathbf{b}'' = 0 \tag{B.3}$$

but no point with $b^0 < 0$. Then, from the assumption of Lorentz invariance, we have the following result.

Lemma. All the points of $\mathscr{T}^+$ satisfy the condition

$$b^0 \geq \|\mathbf{b}\| \tag{B.4}$$

We remark that the conditions (B.1) are equivalent to the conditions

$$\begin{gathered} b^k b_k = 0, \qquad b^0 \geq 0, \qquad b^{rs} b_{rs} = 0 \\ \varepsilon_{rsuv} b^{rs} b^{uv} = 0, \qquad p^2 b^i b_k = b^{ri} b_{rk} \end{gathered} \tag{B.5}$$

and therefore are Lorentz invariant. Since also the cone $\mathcal{T}^+$ is Lorentz invariant, it is sufficient to prove our proposition for a special class of points, which generate all the other points by means of suitable Lorentz transformations.

It follows from the condition C that $\mathcal{T}^+$ is the closure of its interior and therefore it is sufficient to analyze its interior points. In this case, (B.4) cannot be an equality and by means of a Lorentz transformation we can always obtain

$$b^0 > 0, \qquad \mathbf{b} = 0, \qquad b^{12} = b^{30} = 0 \tag{B.6}$$

We indicate by $b^\alpha(\zeta)$ the coordinates obtained from b^α by means of a Lorentz boost with rapidity ζ along the axis b^3 and we define the quantities

$$b^\alpha_\pm = \lim_{\zeta \to \pm\infty} \exp(-|\zeta|) b^\alpha(\zeta) \tag{B.7}$$

Since $\mathcal{T}^+$ is Lorentz and dilatation invariant and it is also closed, it contains the points with coordinates $b^\alpha_\pm$. One can easily see that the only nonvanishing limits are

$$\begin{aligned} b^0_\pm &= \tfrac{1}{2} b^0, & b^3_\pm &= \pm \tfrac{1}{2} b^0 \\ b^{10}_\pm &= \tfrac{1}{2}(b^{10} \pm b^{13}), & b^{20}_\pm &= \tfrac{1}{2}(b^{20} \pm b^{23}) \\ b^{23}_\pm &= \tfrac{1}{2}(b^{23} \pm b^{20}), & b^{31}_\pm &= \tfrac{1}{2}(b^{31} \pm b^{01}) \end{aligned} \tag{B.8}$$

and it follows that

$$b^\alpha = b^\alpha_+ + b^\alpha_- \tag{B.9}$$

Note that the coordinates $b^\alpha_\pm$ satisfy the conditions (B.1). All the points which satisfy these conditions with a given value of p are connected by a rotation and a dilatation. It follows that we have only to determine the allowed values of p. If we perform the operations described above starting from a point with the properties (B.3), we see that $p = 0$ is an allowed value. From the convexity and closure assumptions it follows that p has to satisfy a condition of the kind (B.2). The case $l = 0$ is excluded by the condition B and $l = \infty$ is excluded by the condition C.

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